Differential Equations/Math 256
Assignment Solutions

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Problem 1.

Grading notes. Breakdown will be $(3, 4, 3)$, probably itemized as follows:

- 1 point for writing down the correct integral
- 1 point for attempting each integration by parts that should be attempted (1 for parts a,c and 2 for part b)
- 1 point for a correct final answer

Solution.

a) - Given $f(t) = t^2$, the Laplace transform is

$$F(s) = [\mathcal{L}f](s) = \int_0^\infty t^2 e^{-st}\,dt$$

- We evaluate this integral with two rounds of integration by parts:

$$\begin{aligned}
\int_0^\infty t^2 e^{-st}\,dt &= \cancelto{0}{-t^2 s^{-1} e^{-st}\Big|_{t=0}^{t\to\infty}} + 2s^{-1}\int_0^\infty t e^{-st}\,dt \\
&= 2s^{-1}\left[\cancelto{0}{-ts^{-1}e^{-st}\Big|_{t=0}^{t\to\infty}} + s^{-1}\int_0^\infty e^{-st}\,dt\right] \\
&= 2s^{-2}\left[s^{-1}e^{-st}\Big|_{t=0}^{t\to\infty}\right] \\
&= 2s^{-3}
\end{aligned}$$

b) - Given $f(t) = \cos 2t$, the Laplace transform is

$$F(s) = [\mathcal{L}f](s) = \int_0^\infty e^{-st}\cos 2t\,dt$$

- Two rounds of integration by parts will give us an algebraic equation that we can solve for $F(s)$:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st}\cos 2t\,dt \\
&= \cancelto{0}{\frac{1}{2}e^{-st}\sin 2t\Big|_0^\infty} + \frac{s}{2}\int_0^\infty e^{-st}\sin 2t\,dt \\
\int_0^\infty e^{-st}\sin 2t\,dt &= \frac{1}{2}e^{-st}\cos 2t\Big|_0^\infty - \frac{s}{2}\int_0^\infty e^{-st}\cos 2t\,dt \\
&= \frac{1}{2} - \frac{s}{2}F(s)
\end{aligned}$$

- As promised, we get

$$\begin{aligned}
F(s) &= \frac{1}{4}s\left(1 - sF(s)\right) \\
\left(1 + \frac{1}{4}s^2\right)F(s) &= \frac{s}{4} \\
F(s) &= \frac{s}{s^2+4}
\end{aligned}$$

c) • Given $f(t) = te^t$, the Laplace transform is

$$F(s) = [\mathcal{L}f](s) = \int_0^\infty e^{-st} te^t \, dt = \int_0^\infty te^{(1-s)t} \, dt$$

• This transform only requires one application of integration by parts:

$$\begin{aligned}
F(s) &= \int_0^\infty te^{(1-s)t} \, dt \\
&= \cancelto{0}{\frac{1}{1-s} te^{(1-s)t}\Big|_0^\infty} - \frac{1}{1-s} \int_0^\infty e^{(1-s)t} \, dt \\
&= -\frac{1}{1-s} \left(\frac{1}{1-s} e^{(1-s)t}\Big|_0^\infty \right) \\
&= \frac{1}{(s-1)^2}
\end{aligned}$$

Problem 2.

Grading notes. Five points for each part: two for writing down the right integral, three for evaluating it.

Solution.

a) By breaking up the domain of integration, we can compute the transform as follows:

$$\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st} \, dt \\
&= \int_0^1 \cancelto{0}{f(t)} e^{-st} \, dt + \int_1^2 f(t)e^{-st} \, dt + \int_2^\infty \cancelto{0}{f(t)} e^{-st} \, dt \\
&= \int_1^2 te^{-st} \, dt \\
&= -ts^{-1}e^{-st}\Big|_1^2 + s^{-1} \int_1^2 e^{-st} \, dt \\
&= s^{-1}e^{-s} - 2s^{-1}e^{-2s} - s^{-2}e^{-st}\Big|_1^2 \\
&= s^{-1}e^{-s}(1 - 2e^{-2}) + s^{-2}(1 - e^{-s}) \\
&= s^{-2}e^{-s}((s+1) - (2s+1)e^{-s})
\end{aligned}$$

b) We again break up the domain of integration and obtain an equation that we can solve:

$$\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st} \, dt \\
&= -\int_\pi^{2\pi} e^{-st} \sin t \, dt \\
&= -\left[-e^{-st} \cos t\Big|_\pi^{2\pi} - s \int_\pi^{2\pi} e^{-st} \cos t \, dt \right] \\
&= e^{-\pi s} + e^{-2\pi s} + s \left(\cancelto{0}{e^{-st} \sin t\Big|_\pi^{2\pi}} + s \int_\pi^{2\pi} e^{-st} \sin t \, dt \right) \\
&= e^{-\pi s} + e^{-2\pi s} - s^2 F(s) \\
(1+s^2)F(s) &= e^{-\pi s}(1 + e^{-\pi s}) \\
F(s) &= \frac{e^{-\pi s}(1 + e^{-\pi s})}{(1+s^2)}
\end{aligned}$$

Watch out for sign errors! There are lots of opportunities to make them. A good sanity check to do at the end is to notice that the function $-e^{-st} \sin t$ is nonnegative on the interval $[\pi, 2\pi]$, so its integral should be nonnegative, too.

Problem 3.

Grading notes. Breakdown is $(2, 2, 3, 3)$. In each part, one point for denominators, the rest for numerators.

Solution.

a) This one is a warm-up:

$$\frac{s}{s^2-1} = \frac{A}{s-1} + \frac{B}{s+1}$$

b)
- First we factor the denominator as $(s-1)(s+1)(s^2+1)$ and write the rational function as a sum of rational functions, each with one of the factors for its denominator and some undetermined polynomial for its numerator:

$$\frac{1}{(s^2-1)(s^2+1)} = \frac{A(s)}{s-1} + \frac{B(s)}{s+1} + \frac{C(s)}{s^2+1}$$

- When we cross-multiply to get a common denominator, the polynomial $A(s)$ will be multiplied by $(s+1)(s^2+1)$, which tells us that we will have to deal with four powers of s $(1, s, s^2, s^3)$, so the polynomials $A(s)$, $B(s)$, and $C(s)$ have four coefficients total
- Therefore one of them has two coefficients and the other two have one each; the "odd one out" should be the term involving s^2+1, so we let $C(s) = Cs + D$, while $A(s) \equiv A$ and $B(s) \equiv B$
- The decomposition is therefore

$$\frac{1}{(s^2-1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

c)
- Here, we first cancel the factor of s^2, then break the remaining rational function into a sum over the factors of the denominator, remembering to include all powers corresponding to a given root:

$$\begin{aligned}\frac{s^2(s-1)}{s^2(s-4)(s-2)^2} &= \frac{s-1}{(s-4)(s-2)^2} \\ &= \frac{A(s)}{s-4} + \frac{B(s)}{s-2} + \frac{C(s)}{(s-2)^2} \\ &= \frac{A(s)(s-2)^2 + B(s)(s-4)(s-2) + C(s)(s-4)}{(s-4)(s-2)^2}\end{aligned}$$

- As seen in the last line, to recover the original denominator, we can multiply each term by a polynomial of degree at most 2; the numerator can therefore have degree 2, so the three polynomials $A(s)$, $B(s)$, $C(s)$ only need three coefficients between them, and all three are thus constant
- We therefore write

$$\frac{s^2(s-1)}{s^2(s-4)(s-2)^2} = \frac{A}{s-4} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

d)
- Again, we factor the denominator and include terms corresponding to all powers of each irreducible factor, up to the highest power that appears:

$$\frac{s^3-3}{(s^2+5)^2(s+10)^3} = \frac{A(s)}{s^2+5} + \frac{B(s)}{(s^2+5)^2} + \frac{C(s)}{s+10} + \frac{D(s)}{(s+10)^2} + \frac{E(s)}{(s+10)^3}$$

- When we multiply this expansion by the denominator $(s^2+5)^2(s+10)^3$, the highest-order term obtained is $C(s)(s^2+5)^2(s+10)^2$, which has order 6
- We therefore need 7 coefficients, distributed among the 5 terms of the expansion; by the same "odd one out" argument, the sensible thing is to give two coefficients to the numerators in each of the (s^2+5) terms, and one coefficient each to the three other terms, as follows:

$$\frac{s^3-3}{(s^2+5)^2(s+10)^3} = \frac{A_1 s + A_2}{s^2+5} + \frac{B_1 s + B_2}{(s^2+5)^2} + \frac{C}{s+10} + \frac{D}{(s+10)^2} + \frac{E}{(s+10)^3}$$

Problem 4.

Grading notes. Five points per part: three for breaking F into recognizable transforms (via partial fractions in part a, trig separation in part b), two more for finishing with f correct.

Solution.

a)
- Perform the partial fraction decomposition:

$$\begin{aligned}\frac{2s^2+5s+1}{s^2(s^2+4)} &= \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+4}\\ &= \frac{1}{4}\left[\frac{5}{s}+\frac{1}{s^2}+\frac{-5s+7}{s^2+4}\right]\end{aligned}$$

- Notice that scalar multiples of the following Laplace transforms are present:

$$\begin{aligned}\mathcal{L}[1] &= \frac{1}{s}\\ \mathcal{L}[t] &= \frac{1}{s^2}\\ \mathcal{L}[\sin 2t] &= \frac{2}{s^2+4}\\ \mathcal{L}[\cos 2t] &= \frac{s}{s^2+4}\end{aligned}$$

- Use linearity to put together the original function:

$$\begin{aligned}F(s) &= \frac{1}{4}\left[\frac{5}{s}+\frac{1}{s^2}+\frac{-5s+7}{s^2+4}\right]\\ &= \frac{1}{4}\left[5\left(\frac{1}{s}\right)+\frac{1}{s^2}-5\left(\frac{s}{s^2+4}\right)+\frac{7}{2}\left(\frac{2}{s^2+4}\right)\right]\\ &= \frac{1}{4}\left[5\mathcal{L}[1]+\mathcal{L}[t]-5\mathcal{L}[\cos 2t]+\frac{7}{2}\mathcal{L}[\sin 2t]\right]\\ &= \mathcal{L}\left[\frac{1}{4}\left(5+t-5\cos 2t+\frac{7}{2}\sin 2t\right)\right]\\ f(t) &= \frac{1}{4}\left(5+t-5\cos 2t+\frac{7}{2}\sin 2t\right)\end{aligned}$$

b)
- Partial fractions will not help us here: the denominator has no real roots (its roots are $3 \pm i$)
- Instead, when we see a linear numerator on top of a quadratic denominator that can't be factored usefully, we should immediately rearrange things to find the Laplace transforms given in the hint (and for which we calculated special cases in problems 1 and 2):

$$\begin{aligned}F(s) &= \frac{8s-22}{s^2-6s+10}\\ &= \frac{8s-22}{(s-3)^2+1}\\ &= \frac{8(s-3)+2}{(s-3)^2+1}\\ &= 8\left(\frac{s-3}{(s-3)^2+1}\right)+2\left(\frac{1}{(s-3)^2+1}\right)\\ &= 8\mathcal{L}[e^{3t}\cos t]+2\mathcal{L}[e^{3t}\sin t]\\ &= \mathcal{L}[e^{3t}(8\cos t+2\sin t)]\\ f(t) &= e^{3t}(8\cos t+2\sin t)\end{aligned}$$

Problem 5.

Grading notes. Breakdown is $(3, 4, 3)$. In each point, two points for Y, the rest for y.

Solution. Throughout this problem, we rely on the following identities, which come from integration by parts:

$$\begin{aligned}\mathcal{L}[y'(t)] &= sY(s) - y(0)\\ \mathcal{L}[y''(t)] &= s^2Y(s) - sy(0) - y'(0)\end{aligned}$$

a)
- First we take the Laplace transform of both sides of the ODE, incorporating the initial data, and find the Laplace transform of the solution:

$$\begin{aligned}\mathcal{L}[y'' - y' - 6y] &= \mathcal{L}[0]\\ s^2Y(s) - sy(0) - y'(0) - [sY(s) - y(0)] - 6Y(s) &= 0\\ (s^2 - s - 6)Y(s) &= sy(0) + (y'(0) - y(0))\\ &= s - 2\\ Y(s) &= \frac{s-2}{s^2 - s - 6}\end{aligned}$$

- Now we play with partial fractions and our table of known Laplace transforms to figure out what $y(t)$ must be:

$$\begin{aligned}Y(s) &= \frac{s-2}{(s-3)(s+2)}\\ &= \frac{A}{s-3} + \frac{B}{s+2}\\ &= \frac{1}{5}\left(\frac{1}{s-3}\right) + \frac{4}{5}\left(\frac{1}{s+2}\right)\\ &= \frac{1}{5}\mathcal{L}\left[e^{3t}\right] + \frac{4}{5}\mathcal{L}\left[e^{-2t}\right]\\ &= \mathcal{L}\left[\frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}\right]\\ y(t) &= \frac{1}{5}(e^{3t} + 4e^{-2t})\end{aligned}$$

b)
- Same routine, take Laplace transforms of both sides and find the transform of the solution:

$$\begin{aligned}\mathcal{L}[4y'' + 3y'] &= \mathcal{L}[4]\\ 4[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] &= \frac{4}{s}\\ (4s^2 + 3s)Y(s) - (4sy(0) + 4y'(0) + 3y(0)) &= \frac{4}{s}\\ s(4s+3)Y(s) &= \frac{4}{s} + 4sy(0) + 4y'(0) + 3y(0)\\ &= \frac{4}{s} - 8s - 18\\ &= -2\left[\frac{4s^2 + 9s - 2}{s}\right]\\ Y(s) &= -2\left[\frac{4s^2 + 9s - 2}{s^2(4s+3)}\right]\end{aligned}$$

- Play with partial fractions until you get a linear combination of Laplace transforms that you recognize:

$$\begin{aligned}Y(s) &= -2\left[\frac{A}{s} + \frac{B}{s^2} + \frac{C}{4s+3}\right]\\ As(4s+3) + B(4s+3) + Cs^2 &= (4A + C)s^2 + (3A + 4B)s + 3B\\ &= 4s^2 + 9s - 2\\ Y(s) &= -2\left[\frac{35}{9}\left(\frac{1}{s}\right) - \frac{2}{3}\left(\frac{1}{s^2}\right) - \frac{26}{9}\left(\frac{1}{s + 3/4}\right)\right]\\ y(t) &= \frac{2}{9}\left(6t - 35 + 26e^{-3t/4}\right)\end{aligned}$$

- Yes, the numbers are weird, but you can check by differentiating. It's experiences like this that build character.

c) • Once again, take Laplace transforms and rearrange:

$$\begin{aligned}
\mathcal{L}[y'' - 2y' + 2y] &= \mathcal{L}[\cos t] \\
[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 2Y(s) &= \frac{s}{s^2+1} \\
(s^2 - 2s + 2)Y(s) - (s-2) &= \frac{s}{s^2+1} \\
Y(s) &= \frac{s^3 - 2s^2 + 2s - 2}{(s^2+1)(s^2-2s+2)}
\end{aligned}$$

• Again, play with partial fractions:

$$\begin{aligned}
Y(s) &= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2} \\
&= \frac{1}{5}\left(\frac{s}{s^2+1}\right) - \frac{2}{5}\left(\frac{1}{s^2+1}\right) + \frac{4}{5}\left(\frac{s-1}{(s-1)^2+1}\right) - \frac{2}{5}\left(\frac{1}{(s-1)^2+1}\right) \\
&= \frac{1}{5}\mathcal{L}[\cos t] - \frac{2}{5}\mathcal{L}[\sin t] + \frac{4}{5}\mathcal{L}[e^t\cos t] - \frac{2}{5}\mathcal{L}[e^t \sin t] \\
y(t) &= \frac{1}{5}\left[\cos t - 2\sin t + 2e^t(2\cos - \sin t)\right]
\end{aligned}$$

Problem 6.

Grading notes. Breakdown: $(2, 3, 2, 3)$. One point for the graph, the rest for the step function expression and transform.

Solution. See attached sheet for graph sketches. Throughout this problem we use the fact that $\mathcal{L}[H(t-c)f(t-c)] = e^{-cs}F(s)$, which like all good things in life and Laplace transforms, comes from integration by parts.

a) • We can write the function as

$$f(t) = (t-2)^2 H(t-2)$$

• This lets us compute the Laplace transform:

$$\begin{aligned}
F(s) &= e^{-2s}\mathcal{L}[t^2] \\
&= 2s^{-3}e^{-2s}
\end{aligned}$$

b) • We start by writing

$$f(t) = (t^2 - 2t + 2)H(t-1)$$

• We need to write this as $g(t-1)H(t-1)$ for some function g, which we find as follows:

$$\begin{aligned}
f(t) &= ((t-1)^2 + 1)H(t-1) \\
&= g(t-1)H(t-1) \quad \text{where } g(t) = t^2 + 1
\end{aligned}$$

• Now we take the transform:

$$\begin{aligned}
\mathcal{L}[g(t-1)H(t-1)] &= e^{-s}\mathcal{L}[t^2+1] \\
&= s^{-3}e^{-s}(s^2+2)
\end{aligned}$$

c) If the graph of this function was a staircase, you could climb it in two steps, so the function is a sum of two step functions, and its Laplace transform is immediate:

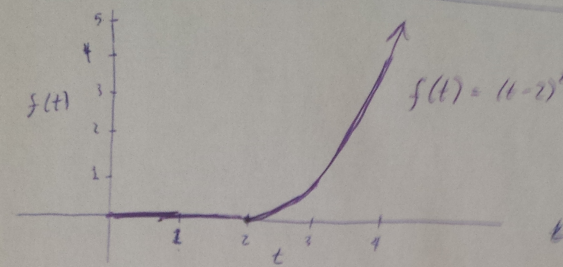
$$\begin{aligned}
f(t) &= H(t-1) + H(t-2) \\
F(s) &= \mathcal{L}[H(t-1) + H(t-2)] \\
&= s^{-1}e^{-s}(1 + e^{-s})
\end{aligned}$$

d) We write down the most literal-minded step-function interpretation of the piecewise definition given for the function f, then collect like terms and take the Laplace transform for fun and profit:

$$\begin{aligned}
f(t) &= t(1 - H(t-1)) + H(t-1) \\
&= t - (t-1)H(t-1) \\
F(s) &= \mathcal{L}[t] - \mathcal{L}[(t-1)H(t-1)] \\
&= (1 - e^{-s})\mathcal{L}[t] \\
&= s^{-2}(1 - e^{-s})
\end{aligned}$$

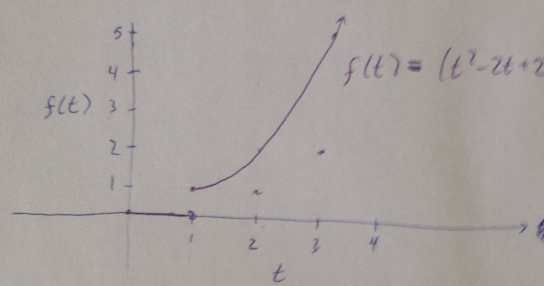
Problem #6: Graph sketches

a)



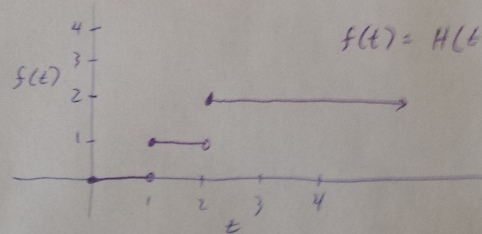
$f(t) = (t-2)^2 H(t-2)$

b)



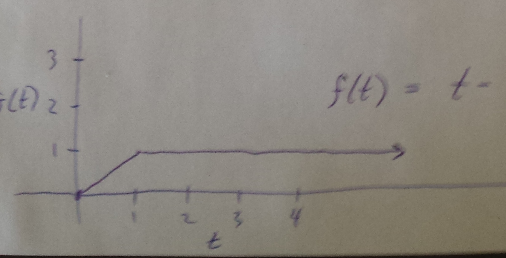
$f(t) = (t^2-2t+2)H(t-1)$

c)



$f(t) = H(t-1) + H(t-2)$

d)



$f(t) = t - (t-1)H(t-1)$

Problem 7.

Grading notes. Breakdown is $(3, 2, 3, 2)$. Full marks for a correct answer; partial marks for progress, with specific 1- and 2-point answers to be determined as I examine the papers.

Solution.

a) • The given transform resembles one that we already know:

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\, dt = \frac{n!}{s^{n+1}}$$

• To modify this known result for our current problem, we can just replace s with $s-1$, then rearrange:

$$\begin{aligned} F(s) &= \frac{2}{(s-1)^3} \\ &= \int_0^\infty t^2 e^{-(s-1)t}\, dt \\ &= \int_0^\infty (t^2 e^t) e^{-st}\, dt \\ &= \mathcal{L}[t^2 e^t] \\ f(t) &= t^2 e^t \end{aligned}$$

b) • The key here is to recognize the presence of the Heaviside function and figure out the rest with partial fractions:

$$\begin{aligned} F(s) &= \frac{e^{-s}}{s^2+s-2} \\ &= e^{-s}\left[\frac{1}{s^2+s-2}\right] \\ &= \frac{1}{3}e^{-s}\left[\frac{1}{s-1} - \frac{1}{s+2}\right] \\ &= \frac{1}{3}e^{-s}\mathcal{L}[e^t - e^{-2t}] \\ &= \frac{1}{3}\mathcal{L}[(e^{(t-1)} - e^{-2(t-1)})H(t-1)] \\ f(t) &= \frac{1}{3}(e^{(t-1)} - e^{-2(t-1)})H(t-1) \end{aligned}$$

c) • We rewrite the given transform in terms of partial fractions and apply the same trick as in part b, collecting two terms and a constant into a hyperbolic trig function:

$$\begin{aligned} F(s) &= \frac{1}{2}e^{-2s}\left[\frac{1}{s-2} - \frac{1}{s+2}\right] \\ &= \frac{1}{2}e^{-2s}\mathcal{L}[e^{2t} - e^{-2t}] \\ &= e^{-2s}\mathcal{L}[\sinh t] \\ &= \mathcal{L}[\sinh(t-2)H(t-2)] \\ f(t) &= \sinh(t-2)H(t-2) \end{aligned}$$

d) • We recognize this as the transform of a sum of step functions:

$$\begin{aligned} F(s) &= (e^{-s} + e^{-2s} - e^{-3s} - e^{-4s})\frac{1}{s} \\ &= \mathcal{L}[H(t-1) + H(t-2) - H(t-3) - H(t-4)] \\ f(t) &= H(t-1) + H(t-2) - H(t-3) - H(t-4) \\ &= \begin{cases} 0 & t < 1 \\ 1 & 1 \le t < 2 \\ 2 & 2 \le t < 3 \\ 1 & 3 \le t < 4 \\ 0 & t \ge 4 \end{cases} \\ &= \left[\mathbf{1}_{[1,4)} + \mathbf{1}_{[2,3)}\right](t) \end{aligned}$$